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Efficient Detection of Exchangeable Factors in Factor Graphs

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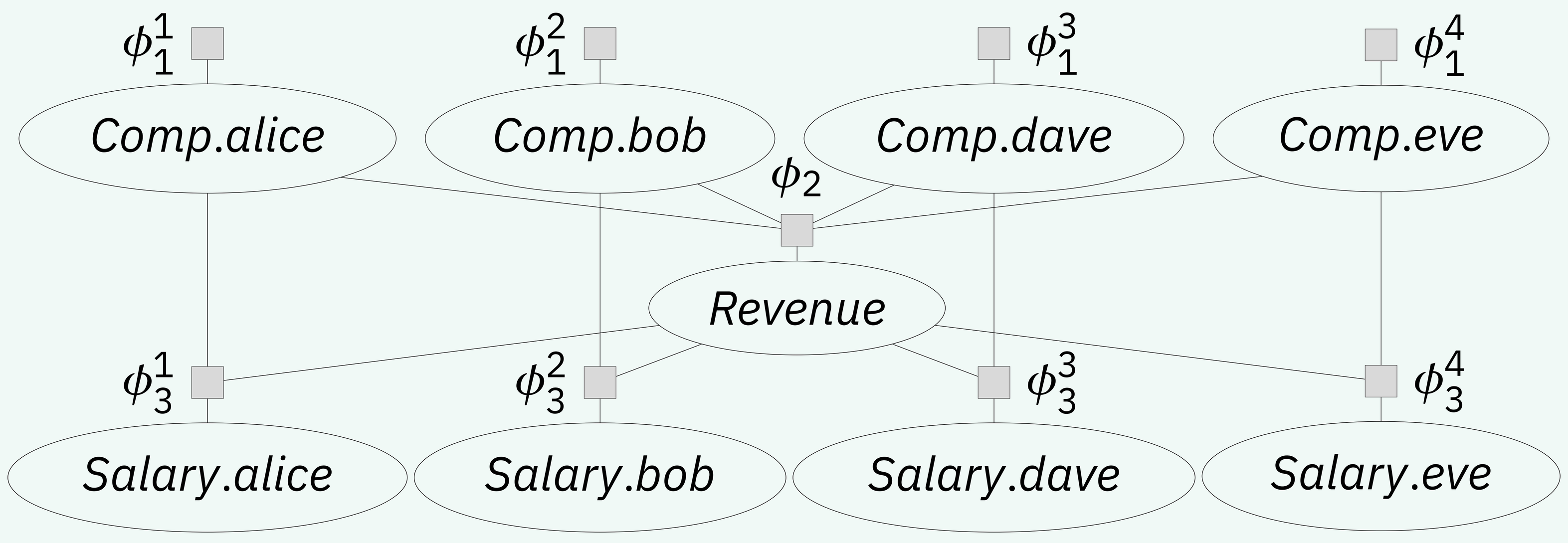
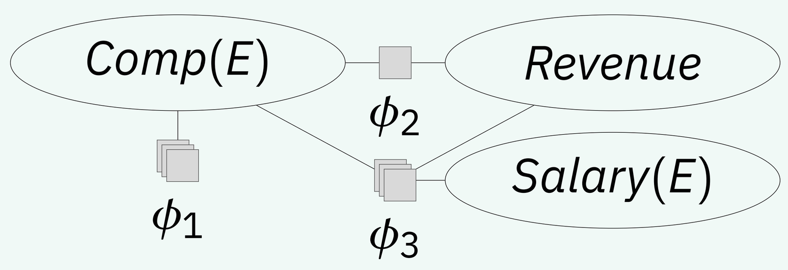
1. Motivation

- Factor graphs compactly encode a probability distribution
- Semantics of a factor graph G over a set of factors $\boldsymbol{\Phi}$:

$$P_G = \frac{1}{Z} \prod_{\phi \in \boldsymbol{\Phi}} \phi$$

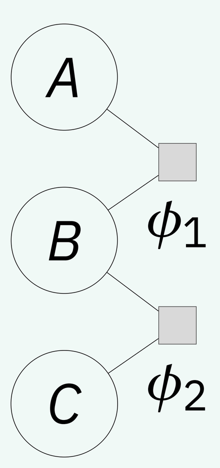


- Parametric factor graphs introduce logical variables to represent groups of random variables
- Parametric factor graphs enable lifted inference (idea: exploit indistinguishability of objects using exponentiation)



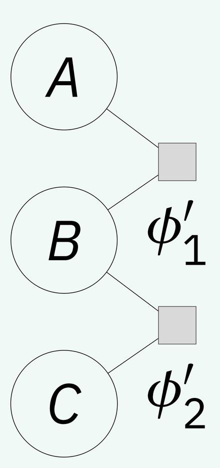
2. Problem Setup

- Goal: Efficiently detect exchangeable factors to group them



A	B	$\phi_1(A,B)$
true	true	φ_1
true	false	φ_2
false	true	φ_3
false	false	φ_4

C	B	$\phi_2(C,B)$
true	true	φ_1
true	false	φ_2
false	true	φ_3
false	false	φ_4

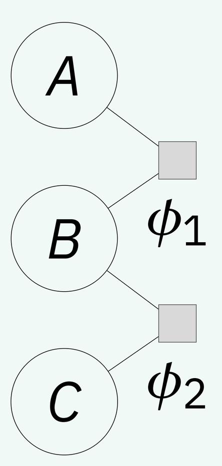


A	B	$\phi'_1(A,B)$
true	true	φ_1
true	false	φ_2
false	true	φ_3
false	false	φ_4

B	C	$\phi'_2(B,C)$
true	true	φ_1
true	false	φ_3
false	true	φ_2
false	false	φ_4

3. Buckets

- Buckets count occurrences of range values in an assignment



A	B	$\phi_1(A,B)$	b
true	true	φ_1	$[2,0]$
true	false	φ_2	$[1,1]$
false	true	φ_3	$[1,1]$
false	false	φ_4	$[0,2]$

B	C	$\phi_2(B,C)$	b
true	true	φ_1	$[2,0]$
true	false	φ_3	$[1,1]$
false	true	φ_2	$[1,1]$
false	false	φ_4	$[0,2]$

b	$\phi_1(b)$	$\phi_2(b)$
$[2,0]$	$\{\varphi_1\}$	$\{\varphi_1\}$
$[1,1]$	$\{\varphi_2,\varphi_3\}$	$\{\varphi_3,\varphi_2\}$
$[0,2]$	$\{\varphi_4\}$	$\{\varphi_4\}$

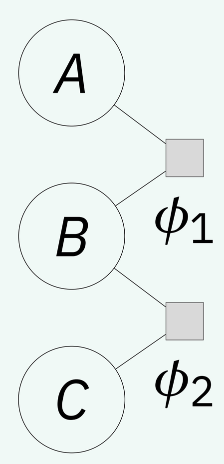
b	$\phi_1^{\succ}(b)$	$\phi_2^{\succ}(b)$
$[2,0]$	$\langle\varphi_1\rangle$	$\langle\varphi_1\rangle$
$[1,1]$	$\langle\varphi_2,\varphi_3\rangle$	$\langle\varphi_3,\varphi_2\rangle$
$[0,2]$	$\langle\varphi_4\rangle$	$\langle\varphi_4\rangle$

Properties of buckets:

- If at least one bucket is mapped to different multisets of values by ϕ_1 and ϕ_2, then ϕ_1 and ϕ_2 are not exchangeable
- Two factors are exchangeable if and only if there exists a permutation of their arguments such that each bucket is mapped to the same ordered multiset of values

4. Detection of Exchangeable Factors (DEFT) Algorithm

- Iterate over buckets and search for possible rearrangements to obtain identically ordered multisets within each bucket



A	B	$\phi_1(A,B)$	b
true	true	φ_1	$[2,0]$
true	false	φ_2	$[1,1]$
false	true	φ_3	$[1,1]$
false	false	φ_4	$[0,2]$

B	C	$\phi_2(B,C)$	b
true	true	φ_1	$[2,0]$
true	false	φ_3	$[1,1]$
false	true	φ_2	$[1,1]$
false	false	φ_4	$[0,2]$

b	$\phi_1^{\succ}(b)$	$\phi_2^{\succ}(b)$
$[2,0]$	$\langle\varphi_1\rangle$	$\langle\varphi_1\rangle$
$[1,1]$	$\langle\varphi_2,\varphi_3\rangle$	$\langle\varphi_3,\varphi_2\rangle$
$[0,2]$	$\langle\varphi_4\rangle$	$\langle\varphi_4\rangle$

- Possible rearrangements:
  - Bucket $[2,0]$: $B \to \{1,2\}$, $C \to \{1,2\}$
  - Bucket $[1,1]$: $B \to \{2\}$, $C \to \{1\}$
  - Bucket $[0,2]$: $B \to \{1,2\}$, $C \to \{1,2\}$
  - Intersection over all buckets: $B \to \{2\}$, $C \to \{1\}$
- Result: Locate B at position 2 and C at position 1 in ϕ_2 to ensure identical potential tables of ϕ_1 and ϕ_2

5. Theoretical Guarantees

- Duplicate potentials in buckets increase complexity
- Degree of freedom of a bucket b:

$$\mathscr{F}(b) = \prod_{\varphi \in \text{unique}(\phi^{\succ}(b))} \text{count}(\phi^{\succ}(b), \varphi)!$$

- Degree of freedom of a factor ϕ:

$$\mathscr{F}(\phi) = \min_{b \in \{b \mid b \in \mathscr{B}(\phi) \wedge |\phi^{\succ}(b)| > 1\}} \mathscr{F}(b)$$

b	$\phi_1'^{\succ}(b)$	$\phi_2'^{\succ}(b)$
$[3,0]$	$\langle\varphi_1\rangle$	$\langle\varphi_1\rangle$
$[2,1]$	$\langle\varphi_2,\varphi_3,\varphi_5\rangle$	$\langle\varphi_3,\varphi_5,\varphi_2\rangle$
$[1,2]$	$\langle\varphi_4,\varphi_6,\varphi_6\rangle$	$\langle\varphi_6,\varphi_4,\varphi_6\rangle$
$[0,3]$	$\langle\varphi_7\rangle$	$\langle\varphi_7\rangle$

- The number of table comparisons needed to check whether ϕ_1 and ϕ_2 are exchangeable is in $O(d)$, where $d = \min\{\mathscr{F}(\phi_1), \mathscr{F}(\phi_2)\}$ (i.e., d is factorial)
- In many practical settings it holds that $d \ll n!$
- The degree of freedom is upper-bounded by $n!$

6. Experiments

- Comparison of run times of DEFT, ACP (previous work), and a »naive« approach
- Average over exchangeable and non-exchangeable factors with a proportion of identical potentials in $\{0.0, 0.1, 0.2, 0.5, 0.8, 0.9, 1.0\}$
- Timeout after 30 minutes per instance